

A new class of invariants in the lepton sector

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Abstract

We construct a new set of combinations from the mass matrices of the charged leptons and neutrinos that are invariant under basis transformation, hereafter *the* invariants. We use these invariants to study various symmetries and neutrino mass textures in a basis independent way. In particular, we show that by using these invariants the ansatz such as $\mu - \tau$ exchange and reflection symmetries, various texture zeros and flavor symmetries can be expressed in a general basis.

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1 Introduction

There is a consensus among neutrino physicists that the recent neutrino data from the solar and atmospheric neutrino observations [1], the KamLAND reactor experiment [2] and the long baseline experiments [3] can be explained only through neutrino oscillation. As is well-known the oscillation scenario is based on the fact that the neutrinos are massive and do mix. The three by three mass matrix of the active neutrinos, m_ν , introduces new parameters to the standard model. Among these parameters, six parameters in principle show up in the neutrino oscillation probabilities: two mass-square splittings, three mixing angles and one Dirac CP-violating phase. To this list, one should add the mass scale of neutrinos (*i.e.*, the mass of the lightest neutrino) which does not appear in the oscillation probability and has to be derived in other types of experiments such as the beta decay experiments or the cosmological observations.

So far the nature of neutrinos (Majorana vs. Dirac) is not known. However, the majority of neutrino mass models predict a Majorana type neutrino mass matrix at low energies. Throughout this paper, we shall assume that neutrinos are of Majorana type which implies that their mass matrix is symmetric. If neutrinos are of Majorana type, in addition to the above parameters the mass matrix will contain two more physical degrees of freedom: two more CP-violating phases which are called Majorana phases. These two phases do not appear in the oscillation probabilities. Even if neutrinos are proved to be of Majorana type, extracting the Majorana phases is going to be quite challenging if possible at all [4]. Moreover, only a combination of the phases can be measured. That is, separately extracting each of the Majorana phases will not be possible with the present methods.

Sources of CP-violation are associated with the phases of the neutrino mass matrix; however, one has to be aware that by rephasing the neutrino fields the phases of the elements of the mass matrix also change. Since the seminal work by Jarlskog [5], defining invariants under field rephasing and basis transformation has proved to be very useful for studying the CP-violation in both quark and lepton sectors. An incomplete list of the papers that have attempted to study the CP-violation in the lepton sector by defining invariants is [6, 7, 8, 9, 10]. As is well-known, because of the presence of the extra CP-violating phases,

the number of the independent invariants is more than what we have in the quark sector (or for the case of Dirac neutrinos). Recently, the necessary and sufficient conditions for CP-violation has been systematically formulated in terms of the rephasing invariants in the mass basis of the charged leptons [8, 9]. It is also possible to study the CP-violation in terms of the combinations of neutrino and charged lepton mass matrices that are invariant under general basis transformation [7]. In this paper, we introduce a new class of invariants under general basis transformation. As we shall see, this new set of invariants is very helpful for studying the symmetries of the neutrino mass matrix.

Among the nine parameters of the neutrino mass matrix, the two mass-square splittings and two of mixing angles are so far measured. There are various running and planned experiments as well as proposals to measure the remaining parameters. However, as alluded to before, even in the most optimistic case, with the present experiments and proposals, we will not be able to extract all the neutrino parameters [4]. Motivated by this fact various theoretical conjectures have been made to reconstruct the neutrino mass matrix. Most of these conjectures are based on symmetries that are apparent only in a particular basis. Examples are texture zeros, $\mu - \tau$ exchange and reflection symmetries [8, 11] and various flavor symmetries [12]. All these symmetries are defined in the mass basis of the charged leptons. Using the invariants defined in the present paper, we can formulate these symmetries in a basis-independent way.

This paper is organized as follows. In sect. 2, we introduce a new class of combinations that are invariant under general basis transformation. In sect. 3, we use the invariants for formulating the symmetries of the neutrino mass matrix in a basis-independent way. We specially discuss the $\mu - \tau$ exchange and reflection symmetries, flavor symmetry conserving $L_\mu - L_\tau$ charge and texture zero ansatz. A summary of results is given in sect. 4.

2 New class of invariants

Consider the following transformation on the charged leptons and neutrinos:

$$\begin{aligned}\ell_{L\alpha} &\rightarrow U_{\alpha\beta} \ell_{L\beta}, \\ \nu_{L\alpha} &\rightarrow U_{\alpha\beta} \nu_{L\beta}, \\ \ell_{R\alpha} &\rightarrow V_{\alpha\beta} \ell_{R\beta},\end{aligned}\tag{1}$$

where U and V are arbitrary unitary matrices and α and β are the flavor indices. The mass term of charged leptons and the effective mass term for Majorana neutrinos at the low energy are of the form:

$$-\mathcal{L} = (m_\ell)_{\alpha\beta} \overline{\ell_{R\alpha}} \ell_{L\beta} + \frac{1}{2} (m_\nu)_{\alpha\beta} \overline{(\nu_{L\alpha})^c} \nu_{L\beta} + \text{H.c.}\tag{2}$$

In order for the Lagrangian to remain invariant under transformations shown in Eq. (1), the mass matrices have to transform as follows

$$\begin{aligned}m_\ell &\rightarrow V m_\ell U^\dagger, \\ m_\nu &\rightarrow U^* m_\nu U^\dagger.\end{aligned}\tag{3}$$

(Notice that we have here used the assumption that neutrinos are of Majorana nature. For Dirac neutrinos with $\nu_{R\alpha} \rightarrow W_{\alpha\beta} \nu_{R\beta}$, m_ν would transform as $W m_\nu U^\dagger$ so all the following discussion should have been reconsidered.) It can be readily shown that under transformations in Eq. (1)

$$\begin{aligned}m_\nu (m_\ell^\dagger m_\ell)^n &\rightarrow U^* m_\nu (m_\ell^\dagger m_\ell)^n U^\dagger, \\ (m_\ell^T m_\ell^*)^m m_\nu &\rightarrow U^* (m_\ell^T m_\ell^*)^m m_\nu U^\dagger,\end{aligned}\tag{4}$$

where m and n are arbitrary integer numbers. In general, a linear combination of these combinations also transforms in the same way:

$$\sum_i [a_i m_\nu (m_\ell^\dagger m_\ell)^{n_i} + b_i (m_\ell^T m_\ell^*)^{m_i} m_\nu] \rightarrow U^* \sum_i [a_i m_\nu (m_\ell^\dagger m_\ell)^{n_i} + b_i (m_\ell^T m_\ell^*)^{m_i} m_\nu] U^\dagger,\tag{5}$$

where a_i and b_i are arbitrary constants. In the above relation m_i and n_i can take positive as well as negative integer numbers. Thus, the determinant of this matrix will transform into itself times $\text{Det}[U^\dagger]\text{Det}[U^*]$, which is a pure phase. As a result, the ratio of any pair of such

determinants is invariant under the transformations shown in Eq. (1):

$$\frac{\text{Det}[\sum_i (a_i m_\nu (m_\ell^\dagger m_\ell)^{n_i} + b_i (m_\ell^T m_\ell^*)^{m_i} m_\nu)]}{\text{Det}[\sum_i (a'_i m_\nu (m_\ell^\dagger m_\ell)^{n'_i} + b'_i (m_\ell^T m_\ell^*)^{m'_i} m_\nu)]} \text{ is invariant.} \quad (6)$$

Moreover,

$$\text{Det} \left[\sum_i (a_i m_\nu (m_\ell^\dagger m_\ell)^{n_i} + b_i (m_\ell^T m_\ell^*)^{m_i} m_\nu) \right] \left(\text{Det} \left[\sum_i (a'_i m_\nu (m_\ell^\dagger m_\ell)^{n'_i} + b'_i (m_\ell^T m_\ell^*)^{m'_i} m_\nu) \right] \right)^*$$

is also invariant. Notice that $m_\nu^n \equiv m_\nu (m_\nu^\dagger m_\nu)^{n-1}$ transforms exactly in the same form as m_ν under basis transformations (*see*, Eqs. (1,3)). As a result, if we replace any of m_ν appearing in Eq. (6) with m_ν^n , the combination will maintain its invariance under transformation (1). The above combinations present an infinite number of invariants. However the 3×3 Majorana neutrino mass matrix contains nine degrees of freedom; so all of these invariants cannot be independent. It is straightforward to show that there is a set of invariants which all the other invariants can be written in terms of them. We will come back to this point at the end of this section. In the following we give a concrete example for such a “complete” set of invariants. We will amply use these invariants in formulating the symmetries of the neutrino mass matrix in sect. 3.

Let us define the following combination of mass matrices:

$$\mathcal{P}_1 \equiv \frac{m_\ell^{-2}}{\text{tr}[m_\ell^{-2}]}, \quad \mathcal{P}_3 \equiv \frac{m_\ell^\dagger m_\ell}{\text{tr}[m_\ell^\dagger m_\ell]}, \quad \mathcal{P}_2 \equiv \mathbf{I} - \mathcal{P}_1 - \mathcal{P}_3, \quad (7)$$

where \mathbf{I} is the three by three identity matrix and $m_\ell^{-2} \equiv m_\ell^{-1} (m_\ell^{-1})^\dagger$. It is straightforward to show that when the eigenvalues of $m_\ell^\dagger m_\ell$ are hierarchical, \mathcal{P}_i act as projection operators. In this section we will not use this property, but this feature will play an important roll in the discussion of sect. 3.

Now using these operators, let us define

$$\begin{aligned} \mathcal{E}_1 &\equiv \frac{\text{Det}[(\mathcal{P}_2 + \mathcal{P}_3)m_\nu - m_\nu \mathcal{P}_1]}{\text{Det}[m_\nu]}, \\ \mathcal{E}_2 &\equiv \frac{\text{Det}[(\mathcal{P}_1 + \mathcal{P}_3)m_\nu - m_\nu \mathcal{P}_2]}{\text{Det}[m_\nu]}, \\ \mathcal{E}_3 &\equiv \frac{\text{Det}[(\mathcal{P}_1 + \mathcal{P}_2)m_\nu - m_\nu \mathcal{P}_3]}{\text{Det}[m_\nu]}, \end{aligned} \quad (8)$$

and

$$\begin{aligned}\mathcal{F}_1 &\equiv \frac{\text{Det}[(\mathcal{P}_1 + \mathcal{P}_3)m_\nu - m_\nu \mathcal{P}_3]}{\text{Det}[m_\nu]}, \\ \mathcal{F}_2 &\equiv \frac{\text{Det}[(\mathcal{P}_2 + \mathcal{P}_3)m_\nu - m_\nu \mathcal{P}_3]}{\text{Det}[m_\nu]}, \\ \mathcal{F}_3 &\equiv \frac{\text{Det}[(\mathcal{P}_2 + \mathcal{P}_3)m_\nu - m_\nu \mathcal{P}_2]}{\text{Det}[m_\nu]}.\end{aligned}\tag{9}$$

It is worth mentioning that only three out of the six combinations defined in Eqs. (8,9) are independent; that is three of them can be written in terms of the other three combinations. It can be shown that the combinations defined in Eq. (6) can be in general written as a combination of three \mathcal{E}_i and \mathcal{F}_i (*e.g.*, \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{F}_1).

Let us define invariants \mathcal{G}_i and \mathcal{H}_i by replacing $\text{Det}[m_\nu]$ in the denominators of Eqs. (8,9) with $\text{Det}[m_\nu^3]$. That is

$$\mathcal{G}_i \equiv \frac{\text{Det}[m_\nu]}{\text{Det}[m_\nu^3]} \mathcal{E}_i \quad \text{and} \quad \mathcal{H}_i \equiv \frac{\text{Det}[m_\nu]}{\text{Det}[m_\nu^3]} \mathcal{F}_i.\tag{10}$$

It can be shown that four out of the six invariants \mathcal{G}_i and \mathcal{H}_i are independent. Depending on the problem in hand, one can choose four independent invariants out of all these twelve invariants to perform various analyses; *for example*, $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{G}_1\}$. An example of the application of such “complete” set of invariants is formulating CP symmetry in the lepton sector. Both in the quark and lepton sectors, using the Jarlskog invariant (\mathcal{J}) to test CP-violation is an established and widely used technique [5]. As is well-known, the Majorana neutrino mass matrix contains more than one source of CP-violation and therefore more than one invariant will be necessary to check CP-invariance [6, 7, 8, 9]. Suppose we take a “complete” set of invariants for testing the CP-violation. If any of these independent invariants is complex, the lepton sector is not CP-invariant (*i.e.*, at least one of the three possible CP-violating phases is different from zero). One may ask whether the opposite is also correct. That is if all these invariants are real, can we conclude that CP is conserved? Since the equations are non-linear, if we take only three invariants, we may in general find a specific solution in addition to the trivial CP-invariant one. However, this specific solution may not be compatible with the neutrino data. Once we examine the realness of the fourth independent invariant, this solution can be excluded.

In the rest of this paper, we show how the combinations \mathcal{E}_i and \mathcal{F}_i can facilitate the basis independent analysis of the symmetries of neutrino sector.

3 Symmetries of neutrino mass matrix in terms of the invariants

There are conjectures (such as $\mu - \tau$ reflection and exchange symmetries [8, 11], various flavor symmetries [12] and texture zero matrices for the neutrino mass matrix [13]) that are customized to be used for the lepton sector. Any model that accommodates such conjectures also has to reproduce the hierarchy of the charged lepton masses. In this section, we use this property to formulate the conjectures in a model-independent fashion.

In the charged lepton mass basis the operators \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 take the following forms

$$\mathcal{P}_1 = \frac{1}{m_e^{-2} + m_\mu^{-2} + m_\tau^{-2}} \begin{pmatrix} m_e^{-2} & 0 & 0 \\ 0 & m_\mu^{-2} & 0 \\ 0 & 0 & m_\tau^{-2} \end{pmatrix}, \quad (11)$$

$$\mathcal{P}_3 = \frac{1}{m_e^2 + m_\mu^2 + m_\tau^2} \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix}, \quad (12)$$

and

$$\mathcal{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \mathcal{P}_1 - \mathcal{P}_3. \quad (13)$$

Using the hierarchy of the charged lepton masses,

$$m_\mu^2/m_\tau^2 = 3.5 \times 10^{-3} \ll m_e^2/m_\mu^2 = 2.3 \times 10^{-5} \ll m_e^2/m_\tau^2 = 7.9 \times 10^{-8},$$

we can write

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}, \alpha^2\right), \quad (14)$$

$$\mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}, \alpha^2\right), \quad (15)$$

$$\mathcal{P}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 - \alpha \end{pmatrix} + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}, \alpha^2\right), \quad (16)$$

where $\alpha \equiv m_\mu^2/m_\tau^2$.

In the charged lepton mass basis, to the first order of the parameter α , \mathcal{E}_i and \mathcal{F}_i defined in Eqs. (8,9) can be written as

$$\begin{aligned} \mathcal{E}_1 &= \frac{m_{ee}(m_{\mu\tau}^2 - m_{\mu\mu}m_{\tau\tau})}{\text{Det}[m_\nu]}, \\ \mathcal{E}_2 &= \frac{m_{\mu\mu}(m_{e\tau}^2 - m_{ee}m_{\tau\tau})}{\text{Det}[m_\nu]}(1 - 4\alpha), \\ \mathcal{E}_3 &= \frac{m_{\tau\tau}(m_{e\mu}^2 - m_{ee}m_{\mu\mu})}{\text{Det}[m_\nu]}(1 - 4\alpha), \end{aligned} \quad (17)$$

and

$$\begin{aligned} \mathcal{F}_1 &= \frac{m_{\mu\tau}(m_{ee}m_{\mu\tau} - m_{e\mu}m_{e\tau})}{\text{Det}[m_\nu]}(1 - 4\alpha), \\ \mathcal{F}_2 &= \frac{m_{e\tau}(m_{\mu\mu}m_{e\tau} - m_{e\mu}m_{\mu\tau})}{\text{Det}[m_\nu]}(1 - 2\alpha), \\ \mathcal{F}_3 &= \frac{m_{e\mu}(m_{e\mu}m_{\tau\tau} - m_{e\tau}m_{\mu\tau})}{\text{Det}[m_\nu]}(1 - 2\alpha), \end{aligned} \quad (18)$$

where $m_{\alpha\beta}$ are the elements of m_ν .

Notice that corrections to these formulae comes from the next to leading order terms which are of the order $\sim 10^{-5}$.

In the subsect. 3.1 we discuss the $\mu - \tau$ symmetries of the neutrino mass matrix and conserved $L_\mu - L_\tau$ flavor symmetry in terms of the invariants. Then, in subsect. 3.2, we discuss the so-called texture zero matrices in an arbitrary basis for the lepton fields.

3.1 $\mu - \tau$ symmetries of m_ν

In this subsection, we study the $\mu - \tau$ reflection and exchange symmetries by using \mathcal{E}_i and \mathcal{F}_i . The aim is to formulate these symmetries in a basis independent way. We first study the

$\mu - \tau$ exchange symmetry, and we then turn our attention to the $\mu - \tau$ reflection symmetry [8, 11].

The $\mu - \tau$ exchange symmetry (the symmetry under $\nu_\mu \leftrightarrow \nu_\tau$) implies

$$m_{e\mu} = m_{e\tau} \quad , \quad m_{\mu\mu} = m_{\tau\tau}, \quad (19)$$

where $m_{\alpha\beta}$ are the entries of m_ν in the charged lepton mass basis. From Eqs. (17,18), these conditions can be expressed in terms of \mathcal{E}_i and \mathcal{F}_i in the following form

$$\left| \frac{\mathcal{F}_2 - \mathcal{F}_3}{\mathcal{F}_2 + \mathcal{F}_3} \right| \ll 1 \quad \text{or equivalently} \quad \left| \frac{\mathcal{E}_2 - \mathcal{E}_3}{\mathcal{E}_2 + \mathcal{E}_3} \right| \ll 1. \quad (20)$$

The above inequalities are basis independent criteria for the $\mu - \tau$ exchange symmetry. In any basis to check for the $\mu - \tau$ exchange symmetry, we can immediately compute the combination in the Eq. (20); if the condition in this equation is satisfied, the neutrino mass matrix is symmetric under the $\mu - \tau$ exchange.

Now let us consider the $\mu - \tau$ reflection symmetry. In the mass basis of charged leptons, the symmetry under $\nu_\mu \leftrightarrow \nu_\tau^*$, (with proper rephasing) implies

$$m_{e\mu} = m_{e\tau}^* \quad , \quad m_{\mu\mu} = m_{\tau\tau}^* \quad , \quad m_{ee} = m_{ee}^* \quad , \quad m_{\mu\tau} = m_{\mu\tau}^*. \quad (21)$$

It is straightforward to show that these equalities, expressed in terms of \mathcal{E}_i and \mathcal{F}_i , implies the following inequalities

$$\left| \frac{\mathcal{F}_3 - \mathcal{F}_2^*}{\mathcal{F}_3 + \mathcal{F}_2^*} \right| \ll 1, \quad \text{and} \quad \left| \frac{\mathcal{E}_1 - \mathcal{E}_1^*}{\mathcal{E}_1 + \mathcal{E}_1^*} \right| \ll 1. \quad (22)$$

As discussed in sect. 2, the above ratios are invariant under general basis transformation Eq. (1). Thus, we have found some basis independent criteria for testing the $\mu - \tau$ reflection symmetry; if either of the inequalities in Eq. (22) does not hold in a given basis, the neutrino mass matrix is not symmetric under the $\mu - \tau$ reflection.

The relations in Eqs. (20) and (22) are the necessary but *not* sufficient conditions for $\mu - \tau$ exchange symmetry and $\mu - \tau$ reflection symmetry, respectively. If the inequalities in Eqs. (20) or (22) do not hold in a given basis, we can conclude that lepton sector is not symmetric under the corresponding transformations. But the reverse is not correct; that

is in certain very special cases it is possible to satisfy the conditions in Eqs. (20) or (22) without having the corresponding symmetry.

Another symmetry proposed in the literature, which has some common features with the $\mu-\tau$ exchange symmetry, is the flavor symmetry with conserved $L_\mu - L_\tau$ [12]. In the charged lepton mass basis, the conservation of $L_\mu - L_\tau$ implies the following form for the neutrino mass matrix

$$\begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad (23)$$

where “ \times ” means that the corresponding entry is nonzero. In the limit of approximation made in Eqs. (17) and (18), for a mass matrix of the form Eq. (23), the invariants \mathcal{E}_i and \mathcal{F}_i take the following values

$$\mathcal{E}_2 = \mathcal{E}_3 = \mathcal{F}_2 = \mathcal{F}_3 = 0, \quad \mathcal{E}_1 = -1 \quad \text{and} \quad \mathcal{F}_1 = -1 + \alpha. \quad (24)$$

[Corrections to these values are of the order $\sim \mathcal{O}(\alpha^2 \sim 10^{-5})$.] The above criterion for the flavor symmetry conserving $L_\mu - L_\tau$, is a necessary but *not* a sufficient criterion for this symmetry. That is, if a model does not satisfy this criterion in one basis, the model does not respect the $L_\mu - L_\tau$ conserving flavor symmetry; but the reverse is not correct.

Notice that a mass matrix of form Eq. (23) cannot accommodate the present data of neutrino experiments because it predicts two degenerate mass eigenvalues. In order to accommodate the present data, the $L_\mu - L_\tau$ conserving symmetry has to be broken; *i.e.*, $L_\mu - L_\tau$ can be only an approximate symmetry. Thus, in practice the exact equalities, “=”, in Eq. (24) has to be replaced by “ \simeq ”.

3.2 Texture zero matrices in terms of invariants

Among the various conjectures that can be imposed on the neutrino mass matrix, the texture zero ansatz have received more attention in the literature [13, 14]. In these scenarios, certain entries of the neutrino mass matrix in the charged lepton mass basis are conjectured to be equal to zero. Such an assumption can originate from a more fundamental theory or an underlying symmetry [15]. On the other hand, most of the models of the neutrino mass

matrix are built based on some symmetry that is apparent only in a certain basis which may not correspond to the mass basis of the charged leptons. In this section, we show that by using the combinations \mathcal{E}_i and \mathcal{F}_i defined in Eqs. (8,9), we can express the conditions for texture zero in a basis-independent way.

As shown in [13], a neutrino mass matrix m_ν with three or more zero entries is not compatible with the data. Moreover, among the fifteen possible two zero texture mass matrices, only seven of them can be made compatible with the present neutrino data [13]. The two zero textures are labeled A_1 , A_2 , B_1 , B_2 , B_3 , B_4 and C . We have listed them in Table 1. The non-vanishing entries in this Table are denoted by \times . These textures have certain predictions for the values of the neutrino parameters. For example, the A_1 and A_2 textures are compatible with data only for normal hierarchical scheme ($m_1 \ll \sqrt{\Delta m_{atm}^2}$).

From the Eqs. (17,18), we readily observe that if one of the diagonal or off-diagonal elements of the mass matrix vanishes, some of \mathcal{E}_i or \mathcal{F}_i will also go to zero (*e.g.*, $m_{ee} = 0 \implies \mathcal{E}_1 \rightarrow 0$; $m_{\mu\tau} = 0 \implies \mathcal{F}_1 \rightarrow 0$ and so on). Remember that under basis transformation, \mathcal{F}_i and \mathcal{E}_i are invariant. Thus, if an \mathcal{E}_i or an \mathcal{F}_i vanishes in a particular basis, it will vanish in all bases. To be precise, there is a correction of order of $\alpha^2 \sim m_e^2/m_\mu^2 \sim 10^{-5}$ to \mathcal{E}_i and \mathcal{F}_i shown in Eqs. (17,18). As a result, when a mass matrix element vanishes, certain \mathcal{F}_i and \mathcal{E}_i become much smaller than the rest but not exactly zero.

Table 1: Two zero texture mass matrices

Label	Neutrino mass matrix m_ν
A_1	$\begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}$
A_2	$\begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$
B_1	$\begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}$
B_2	$\begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ \times & \times & 0 \end{pmatrix}$
B_3	$\begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}$
B_4	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{pmatrix}$
C	$\begin{pmatrix} \times & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}$

Table 2: Values of \mathcal{E}_i and \mathcal{F}_i (leading order) for textures in Table 1

Label	Values of \mathcal{E}_i and \mathcal{F}_i
A_1	$\mathcal{E}_1 \approx \mathcal{E}_3 \approx \mathcal{F}_1 \approx \mathcal{F}_3 \ll \mathcal{E}_2 \approx \mathcal{F}_2$
A_2	$\mathcal{E}_1 \approx \mathcal{E}_2 \approx \mathcal{F}_1 \approx \mathcal{F}_2 \ll \mathcal{E}_3 \approx \mathcal{F}_3$
B_1	$\mathcal{E}_2 \approx \mathcal{F}_2 \ll \mathcal{E}_1 \approx \mathcal{F}_1, \mathcal{E}_3 \approx \mathcal{F}_3$
B_2	$\mathcal{E}_3 \approx \mathcal{F}_3 \ll \mathcal{E}_1 \approx \mathcal{F}_1, \mathcal{E}_2 \approx \mathcal{F}_2$
B_3	$\mathcal{E}_2 \approx \mathcal{E}_3 \approx \mathcal{F}_2 \approx \mathcal{F}_3 \ll \mathcal{E}_1 \approx \mathcal{F}_1$
B_4	$\mathcal{E}_2 \approx \mathcal{E}_3 \approx \mathcal{F}_2 \approx \mathcal{F}_3 \ll \mathcal{E}_1 \approx \mathcal{F}_1$
C	$\mathcal{E}_2 \approx \mathcal{E}_3 \ll \mathcal{F}_2 \approx \mathcal{F}_3 \approx \mathcal{F}_1 - \mathcal{E}_1 \neq 0$

Each of the texture zeros implies a certain pattern for \mathcal{E}_i and \mathcal{F}_i . For example, for the A_1 texture we find $\mathcal{E}_1 \approx \mathcal{E}_3 \approx \mathcal{F}_1 \approx \mathcal{F}_3 \ll \mathcal{E}_2 \approx \mathcal{F}_2$. The patterns of \mathcal{E}_i and \mathcal{F}_i for the rest of the textures are summarized in Table 2. Here, $A \approx B$ means $|A - B|/|A + B| \lesssim O(\alpha^2 \sim 10^{-5})$. Going to higher orders of α makes the analysis *uselessly* cumbersome, especially that in most models that predict texture zeros the vanishing elements of m_ν receive a small correction (due to running or etc.). Thus, in the following we consider the leading order patterns for \mathcal{E}_i and \mathcal{F}_i . That is to perform the analysis, we will replace “ \approx ” with “ $=$ ” and set \mathcal{E}_i and

\mathcal{F}_i for each pattern that according to Table 2 are much smaller than the rest equal to zero.

These patterns can be considered as a test for the two zero textures. That is, by computing \mathcal{E}_i and \mathcal{F}_i in any given basis, one can check if a certain pattern can be the case. It is obvious that if the pattern associated with a certain texture does not hold, m_ν in the charged lepton mass basis will not have the format of that particular texture. In the following, we explore whether the opposite is also true. The question is as follows. Suppose that a certain pattern of \mathcal{E}_i and \mathcal{F}_i listed in Table 2 is realized. Can we then conclude that m_ν in the charged lepton mass basis has the format of the texture corresponding to that particular pattern? To answer this question we check if the equations listed in Table 2 have any solution compatible with the neutrino data other than the particular texture zero solution corresponding to them. To perform the analysis we use the standard parametrization of the neutrino mass matrix presented by the particle data group [16].

Let us first discuss the A_1 and A_2 textures. Notice that among the textures listed in Table 2, only for the A_1 and A_2 textures we have $\mathcal{F}_1 = \mathcal{E}_1 = 0$. In the following, we first check if, despite $m_{ee}, m_{\mu\tau} \neq 0$, we can have $\mathcal{F}_1 = \mathcal{E}_1 = 0$ and then check for solutions with $m_{ee} = 0, m_{\mu\tau} \neq 0$ and $m_{ee} \neq 0, m_{\mu\tau} = 0$. It is straightforward but rather cumbersome to show that, assuming $m_{ee}, m_{\mu\tau} \neq 0$, the only solution of $\mathcal{F}_1 = \mathcal{E}_1 = 0$ is $m_3 = 0, s_{13} = 0$. (In fact, there is another solution which requires $m_1 = -m_2(c_{12}^2 - c_{12}s_{12}s_{13}\tan\theta_{23}e^{i\delta})/(s_{12}^2 + s_{12}c_{12}s_{13}\tan\theta_{23}e^{i\delta})$ but this is not compatible with neutrino data.) It is straightforward to show that $m_3 = s_{13} = 0$ implies $\mathcal{E}_2, \mathcal{E}_3 \neq 0$ so not all of the conditions for the A_1 and A_2 textures can be fulfilled. Thus, so far we have concluded that if the pattern associated to the A_1 or A_2 textures holds (if $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{F}_1 = \mathcal{F}_3 = 0$ or $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{F}_1 = \mathcal{F}_2 = 0$) at least one of the ee or $\mu\tau$ entries must be nonzero. On the other hand, $m_{ee} = 0$ and $\mathcal{E}_1 = \mathcal{F}_1 = 0$ with $m_{\mu\tau} \neq 0$ implies $m_{e\mu}m_{e\tau} = 0$ which is the condition for the A_1 or A_2 textures. These two textures can be distinguished by computing \mathcal{E}_2 and \mathcal{E}_3 and checking which one vanishes. Finally, $m_{ee} \neq 0, m_{\mu\tau} = 0$ and $\mathcal{E}_1 = \mathcal{F}_1 = 0$ implies $m_{\mu\mu}m_{\tau\tau} = 0$ which is not compatible with the data [13]. In sum, we have proved that the equations $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{F}_1 = \mathcal{F}_3 = 0$ ($\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{F}_1 = \mathcal{F}_2 = 0$) are both necessary and sufficient conditions for the A_1 (A_2) texture provided that the mass matrix accommodates the present neutrino data.

Now let us discuss the B_i textures. As shown in Table 2, the conditions for B_1 are

$\mathcal{E}_2 = \mathcal{F}_2 = 0$, $\mathcal{E}_1 = \mathcal{F}_1$ and $\mathcal{E}_3 = \mathcal{F}_3$. Notice that $\mathcal{E}_2 = \mathcal{F}_2 = 0$ automatically implies $\mathcal{E}_1 = \mathcal{F}_1$ and $\mathcal{E}_3 = \mathcal{F}_3$. Thus to check if the conditions shown in the third row of Table 2 guarantee the format of texture B_1 , it is sufficient to solve $\mathcal{E}_2 = \mathcal{F}_2 = 0$. It can be shown that $\mathcal{E}_2 = \mathcal{F}_2 = 0$ (the conditions for B_1), in addition to $m_{\mu\mu} = m_{e\tau} = 0$, have another solution which yields the following relations

$$\begin{cases} |m_2|^2 - |m_1|^2 &= |m_1|^2 \cos \delta (4s_{13}) / (s_{12}c_{12}) \\ |m_3|^2 - |m_1|^2 &= |m_1|^2 (s_{23}^2 - c_{23}^2) / c_{23}^4 \end{cases} . \quad (25)$$

As a result, $s_{13} \cos \delta / (s_{23}^2 - c_{23}^2) \sim (\Delta m_{sol}^2) / (\Delta m_{atm}^2) \ll 1$. Moreover, the second equation of Eq. (25) can be considered as a lower bound on $|m_1|$; with the present data [17], this equation gives the bound $|m_1| > 0.006$ eV at 3σ . Notice that with the present uncertainties on the neutrino data this solution is still acceptable. Thus, the conditions listed in the third row of Table 2, in addition to texture B_1 have another solution which is compatible with the present neutrino data. Future measurements of the neutrino mass scale [4], θ_{23} and $\text{sgn}(|m_3|^2 - |m_1|^2)$ may enable us to test the second equation in Eq. (25). In particular, the NO ν A [18] and T2K [19] experiments can measure θ_{23} and the absolute value of Δm_{31}^2 with very high accuracy. The accuracy in the measurement of $\sin^2 2\theta_{23}$ can reach 1%. If these experiments establish that θ_{23} is close to maximal, the lower bound on $|m_1|$ will be within the reach of the KATRIN experiment [20]. For relatively large values of θ_{13} (*i.e.*, $s_{13} > 0.05$), more futuristic experiments such as the T2KK setup [21] can help us to solve the octant-degeneracy and derive information on $\text{sgn}(|m_3|^2 - |m_1|^2)$ and δ . Such information makes the solution completely testable. In summary, the conditions listed in the third row of Table 2, in addition to texture B_1 have another solution compatible with the present data. Forthcoming data may exclude this solution.

Now let us study the conditions for texture B_2 which are listed in the fourth row of Table 2. The conditions $\mathcal{E}_3 = \mathcal{F}_3 = 0$ automatically yield $\mathcal{E}_1 = \mathcal{F}_1$ and $\mathcal{E}_2 = \mathcal{F}_2$ so it will be sufficient to study the consequences of $\mathcal{E}_3 = \mathcal{F}_3 = 0$. Similarly to the case of texture B_1 , $\mathcal{E}_3 = \mathcal{F}_3 = 0$ for $m_{e\mu}, m_{\tau\tau} \neq 0$ implies

$$\begin{cases} |m_2|^2 - |m_1|^2 &= -|m_1|^2 \cos \delta (4s_{13}) / (s_{12}c_{12}) \\ |m_3|^2 - |m_1|^2 &= -|m_1|^2 (s_{23}^2 - c_{23}^2) / c_{23}^4 \end{cases} . \quad (26)$$

A discussion similar to the one after Eqs. (25) holds here, too. That is, the conditions

$\mathcal{E}_3 = \mathcal{F}_3 = 0$ other than texture B_2 has another solution which is compatible with the present data but can be excluded by the forthcoming NO ν A [18] and T2K [19] experiments.

Now let us discuss the B_3 and B_4 textures whose necessary conditions are $\mathcal{E}_2 = \mathcal{E}_3 = \mathcal{F}_2 = \mathcal{F}_3 = 0$. From Eqs. (25,26), we readily see that if $\mathcal{E}_2 = \mathcal{E}_3 = \mathcal{F}_2 = \mathcal{F}_3 = 0$, there is no solution with $m_{e\mu}, m_{e\tau}, m_{\mu\mu}, m_{\tau\tau} \neq 0$. That is some of these entries should vanish. Considering the different configurations of vanishing entries, we find that $\mathcal{E}_2 = \mathcal{F}_2 = \mathcal{E}_3 = \mathcal{F}_3 = 0$ implies either B_3 or B_4 .

Finally, let us discuss the condition for texture C whose conditions are listed in the last row of Table 2. Notice that for $m_{\mu\mu}, m_{\tau\tau} \neq 0$, $\mathcal{E}_2 = \mathcal{E}_3 = 0$ automatically yields $\mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_1 - \mathcal{E}_1$. In addition to $m_{\mu\mu} = m_{\tau\tau} = 0$, the equations $\mathcal{E}_2 = \mathcal{E}_3 = 0$ have another solution which implies

$$\begin{aligned} \frac{|m_2|^2 - |m_1|^2}{|m_1|^2} &= \frac{\cos 2\theta_{12}}{s_{12}^4}, \\ \frac{|m_3|^2 - |m_1|^2}{|m_1|^2} &\simeq \frac{\cos^2(2\theta_{23})c_{12}^2}{4s_{23}^4 c_{23}^2 s_{13}^2 s_{12}^2}. \end{aligned} \quad (27)$$

The above relations in turn implies

$$\Delta m_{atm}^2 / \Delta m_{sol}^2 = (s_{12}^2 c_{12}^2 / \cos 2\theta_{12})(1/4s_{23}^4 c_{23}^2)[\cos^2(2\theta_{23})/s_{13}^2] \gg 1.$$

The above relation might be tested by forthcoming measurements. If these experiments do not confirm Eq. (27), the aforementioned solution will be ruled out and the texture C will be the only solution of $\mathcal{E}_2 = \mathcal{E}_3 = 0$.

In sum, we have listed the necessary conditions for different texture zero scenarios in Table 2. We have shown that in the case of texture A_1 and A_2 the conditions listed respectively in the first and second rows of this Table are sufficient to establish these textures. Moreover, in the case of textures B_3 and B_4 , the conditions listed in the Table have no solution compatible with the neutrino data other than these textures. However, the conditions for B_1 , B_2 and C can have another solution which might be ruled out by improving the neutrino data.

4 Summary

In this paper, we have studied the symmetries of the lepton sector in a basis independent way by defining a new class of basis invariants constructed out of the lepton mass matrices.

We have focused on the symmetries of the effective mass matrix of neutrinos at low energies (below the electroweak scale) under the assumption that neutrinos are Majorana particles.

As is well-known, even in the most optimistic case through the present methods and proposals, the neutrino mass matrix cannot be fully reconstructed. Motivated by this observation, various neutrino mass matrix ansatz have been developed in the literature. Most of these conjectures are based on symmetries and conditions that are apparent only in a particular basis. We have shown that by using the invariants defined in this paper such symmetries and conditions can be formulated in a basis independent way. We have in particular focused on the $\mu - \tau$ exchange and reflection symmetries, flavor symmetry with conserved $L_\mu - L_\tau$; and texture zeros. We have demonstrated how our invariants can facilitate testing the conditions defining these ansatz in a general basis.

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